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ORIGINAL PAPER

Bautin bifurcation for the Lengyel–Epstein system

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Abstract In this paper, complete analysis is presented to study Bautin bifurcation for the Lengyel–Epstein System

$$\begin{cases} \frac{du}{dt} = a - u - \frac{4uv}{1+u^2}, \\ \frac{dv}{dt} = \sigma b \left(u - \frac{uv}{1+u^2} \right). \end{cases}$$

Sufficient conditions for a and b are given for the system to demonstrate Bautin bifurcation. By using b and $\alpha = a/5$ as bifurcation parameters and computing the first and second Lyapunov coefficients and performing nonlinear transformation, the normal form with unfolding parameters is derived to obtain the bifurcation diagrams such as Hopf and double limit cycle bifurcations. An example is given to confirm that the system has two limit cycles.

Keywords Lengyel–Epstein System · Bautin bifurcation · First and second Lyapunov coefficients · Normal forms · Hopf and double limit cycle bifurcations

Mathematics Subject Classification 34K18

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1 Introduction

The Lengyel–Epstein System is the following reaction-diffusion equations

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + a - u - \frac{4uv}{1+u^2}, \\ \frac{\partial v}{\partial t} = \sigma \left[\Delta v + b \left(u - \frac{uv}{1+u^2} \right) \right], \end{cases} \tag{1.1}$$

which was derived from the chlorite iodide malonic acid (CIMA) chemical reaction introduced by Lengyel and Epstein [5,6] and can be used to design chemical systems capable of displaying stationary, symmetry breaking reaction diffusion patterns (Turing structures). Here u and v are the concentrations of the active iodide I^- and inhibitor (ClO_2^-) at time t, respectively, a and b are positive parameters related to the feed concentrations; $\sigma > 0$ is a rescaling parameter depending on the concentration of the starch.

A closely related system for this chemical reaction mechanism is the chlorine dioxide-iodine-malonic acid (CDIMA) reaction shown below.

$$\begin{cases} MA{+}I_2 \rightarrow IMA + I + H^+ \\ ClO_2 + I^- \rightarrow ClO_2 + \frac{1}{2} I_2 \\ ClO_2^- + 4I^- + 4H^+ \rightarrow Cl + 2I_2 + 2H_2O \\ S + I_2 + I^- \rightarrow SI^{-3} \end{cases}$$

The first reaction serves as a source of the activator I^-, the second produces the inhibitor chlorite ion, the third shows regeneration of iodine, and the last reaction shows the complex formation between the activator iodide (I^-) and the indicator starch.

In the CDIMA system of reaction, the concentration of Malonic acid (MA), Chloride Dioxide (ClO_2) and Iodine (I_2) displays very little variation and essentially they can be considered constant. Since only the activator iodine ion (I^-) and the inhibitor chlorite ion (ClO_2^-) show wide concentration variation, the system can be approximated by two variables model [2,3,7,8].

In the presence of starch which is used as indictor, the diffusion rate of the activator (I^-) is slower than that of the inhibitor (ClO_2^-). The starch which is much bigger molecule forms a chemical complex with I^- effectively reducing the diffusion rate of I^-. This allows the inhibitor to diffuse faster creating a condition that leads to oscillatory phenomenon. In laboratory conditions, a sample of parameters is taken in the range $0 < a < 35, 0 < b < 8, \sigma = 8$.

In Ni and Tang [10], studied the initial value problem of the corresponding reaction-diffusion model with the no-flux boundary condition. Yi et al. [12] used b as the bifurcation parameter and obtained a critical value b^* of b such that both the ODE and PDE models exhibit a Hopf bifurcation as b crosses b^*. They calculated the first Liapunov coefficient which determines the stability and and direction of the periodic solution bifurcating from the equilibrium point for the ODE. From the view point of Chemistry and Physics, periodic solutions represent the oscilations of the concentrations of I^- and ClO_2^-. However the first Liapunov coefficient can be zero for certain value of $\alpha = \frac{a}{5}$. In this situation, the criteria of the stability of the bifurcating periodic solution and the direction of Hopf bifurcation fails. This makes us to consider Bautin bifurcation and calculate the second Liapunov coefficient which leads to the bifurcation of the second limit cycle. Bautin bifurcation is a so-called 2-dimensional bifurcation of two parametric autonomous ODE systems and has very important applications in many dynamical systems in real-world applications.

The rest of the paper is organized as follows. In Sect. 2, we use the method from Kuznetsov [4] to calculate the first and second Lyapunov coefficients and hence obtain the normal form of Lengyel–Epstein System with the unfolding parameters. In Sect. 3, we give one example to verify our theoretical result and end our investigation with concluding remarks.

2 Bautin bifurcation of the ODE model

Let $\alpha = a/5$. Then $(u^*, v^*) = (\alpha, 1+\alpha^2)$ is the unique equilibrium point of Sys.(1.1). Let $w_1 = u - u^*$, $w_2 = v - v^*$. Then in the absence of diffusion, Sys.(1.1) can be transformed as

$$\begin{cases} \frac{dw_1}{dt} = 4\alpha - w_1 - \frac{4(w_1+\alpha)(w_2+1+\alpha^2)}{1+(w_1+\alpha)^2}, \\ \frac{dw_2}{dt} = \sigma b\left[w_1 + \alpha - \frac{(w_1+\alpha)(w_2+1+\alpha^2)}{1+(w_1+\alpha)^2}\right]. \end{cases} \tag{2.1}$$

or

$$\frac{dw}{dt} = J(\alpha, b)w + G(w, \alpha, b). \tag{2.2}$$

where $w = (w_1, w_2)^T$, $G(w, \alpha, b) = (g_1(w, \alpha, b), g_2(w, \alpha, b))^T$,

$$\begin{aligned} J(\alpha, b) &= \begin{pmatrix} \frac{3\alpha^2-5}{1+\alpha^2} & -\frac{4\alpha}{1+\alpha^2} \\ \frac{2\sigma\alpha^2 b}{1+\alpha^2} & -\frac{\sigma\alpha b}{1+\alpha^2} \end{pmatrix}, \\ g_1(w, \alpha, b) &= \frac{4\alpha(3-\alpha^2)}{(1+\alpha^2)^2}w_1^2 + \frac{4\alpha(\alpha^2-1)}{(1+\alpha^2)^2}w_1w_2 + \frac{4(\alpha^4-6\alpha^2+1)}{(1+\alpha^2)^3}w_1^3 \\ &\quad + \frac{4\alpha(3-\alpha^2)}{(1+\alpha^2)^3}w_1^2w_2 - \frac{4\alpha(5-10\alpha^2+\alpha^4)}{(1+\alpha^2)^4}w_1^4 \\ &\quad + \frac{4(1-6\alpha^2+\alpha^4)}{(1+\alpha^2)^4}w_1^3w_2 + \frac{4(-1+15\alpha^2-15\alpha^4+\alpha^6)}{(1+\alpha^2)^5}w_1^5 \\ &\quad - \frac{4\alpha(5-10\alpha^2+\alpha^4)}{(1+\alpha^2)^5}w_1^4w_2 + \mathcal{O}(|u|^6), \\ g_2(w, \alpha, b) &= \frac{\sigma b}{4}g_1(w, \alpha, b). \end{aligned}$$

The linear part of Sys.(2.1) at (0, 0) is

$$\frac{dw}{dt} = Jw. \tag{2.3}$$

Then the characteristic equation of J is

$$\Delta(\lambda) \equiv \lambda^2 + T\lambda + D = 0, \tag{2.4}$$

where

$$T = \frac{3\alpha^2 - \sigma\alpha b - 5}{1+\alpha^2}, \quad D = \frac{5\sigma\alpha b}{1+\alpha^2}.$$

In this paper, we make the following assumptions

(H1) $\alpha > \frac{\sqrt{15}}{3}$.
(H2) $\frac{5}{3} < \alpha^2 < \frac{27+\sqrt{769}}{4}$.
(H3) $\alpha^2 > \frac{27+\sqrt{769}}{4}$.

Then if (H) holds, J has a pair of purely imaginary roots $\pm\omega_0 i$ when $b = b^* \equiv \frac{3\alpha^2-5}{\alpha\sigma}$ where

$$\omega_0 = \frac{\sqrt{5(3\alpha^4 - 2\alpha^2 - 5)}}{1+\alpha^2}.$$

on the half line

$$l = \left\{ \mu = (b, \alpha) : b = b^*, \alpha > \frac{\sqrt{15}}{3} \right\}.$$

Near l, $\Delta(\lambda) = 0$ has two complex roots $\lambda = \nu + i\omega$ and $\bar{\lambda} = \nu - i\omega$ where

$$\nu(b,\alpha) = \frac{3\alpha^2 - b\alpha\sigma b - 5}{2(1+\alpha^2)}, \quad \omega(b,\alpha) = \frac{\sqrt{20b\alpha(1+\alpha^2)\sigma - (5 - 3\alpha^2 + b\alpha\sigma)^2}}{2(1+\alpha^2)}.$$

Then $\nu(b^*, \alpha) = 0$ and $\omega_0 = \omega(b^*, \alpha)$.

Define $\mu = (b, \alpha)$ and $w = (w_1, w_2)^T$. Note that Sys.(2.1) can be written as

$$u = L(\mu)u + F(u, \mu) \tag{2.5}$$

where

$$L(\mu) = J(b, \alpha),$$
$$F(u,\mu) = \frac{1}{2}B(u,u) + \frac{1}{3!}C(u,u,u) + \frac{1}{4!}D(u,u,u,u) + \frac{1}{5!}E(u,u,u,u,u) + \mathcal{O}(|u|^6).$$

Here

$$B(x, y) = \sum_{j,k=1}^{2} \frac{\partial^2 F(u,\mu)}{\partial u_j \partial u_k}\Big|_{u_1=u_2=0} x_j y_k$$

$$= \frac{4[2\alpha(3-\alpha^2)x_1y_1 + (\alpha^2-1)x_1y_2 + (\alpha^2-1)x_2y_1]}{(1+\alpha^2)^2}\begin{pmatrix}1\\ \frac{b\sigma}{4}\end{pmatrix},$$

$$C(x,y,z) = \sum_{j,k,l=1}^{2} \frac{\partial^3 F(u,\mu)}{\partial u_j \partial u_k \partial u_l}\Big|_{u_1=u_2=0} x_j y_k z_l$$
$$= \frac{8[3(1-6\alpha^2+\alpha^4)x_1y_1z_1+\alpha(3-\alpha^2)x_1y_2z_1+\alpha(3-\alpha^2)x_1y_1z_2]}{(1+\alpha^2)^3}\begin{pmatrix}1\\ \frac{b\sigma}{4}\end{pmatrix},$$

$$D(x,y,z,v) = \sum_{j,k,l,r=1}^{2} \frac{\partial^4 F(u,\mu)}{\partial u_j \partial u_k \partial u_l \partial u_r}\Big|_{u_1=u_2=0} x_j y_k z_l v_r$$
$$= \frac{24}{(1+\alpha^2)^4}[-4\alpha(5-10\alpha^2+\alpha^4)x_1y_1z_1v_1+(1-6\alpha^2+\alpha^4)x_1y_2z_1v_1$$
$$+(1-6\alpha^2+\alpha^4)x_1y_1z_2v_1+(1-6\alpha^2+\alpha^4)x_1y_1z_1v_2$$
$$+(1-6\alpha^2+\alpha^4)x_2y_1z_1v_1]\begin{pmatrix}1\\ \frac{b\sigma}{4}\end{pmatrix},$$

$$E(x,y,z,v,w) = \sum_{j,k,l,r,s=1}^{2} \frac{\partial^5 F(u,\mu)}{\partial u_j \partial u_k \partial u_l \partial u_r \partial u_s}\Big|_{u_1=u_2=0} x_j y_k z_l v_r w_s$$
$$= \frac{96}{(1+\alpha^2)^5}[-4\alpha(5-10\alpha^2+\alpha^4)x_1y_1z_1v_1+(1-6\alpha^2+\alpha^4)x_1y_2z_1v_1$$
$$+(1-6\alpha^2+\alpha^4)x_1y_1z_2v_1+(1-6\alpha^2+\alpha^4)x_1y_1z_1v_2$$
$$+(1-6\alpha^2+\alpha^4)x_2y_1z_1v_1]\begin{pmatrix}1\\ \frac{b\sigma}{4}\end{pmatrix}.$$

2.1 First Lyapunov coefficient

In this subsection, we compute the first and second Lyapunov coefficients by using the formulas in [4].

Near $b = b^*$, define

$$Q(b) = \left(1, \frac{-5+3\alpha^2+b\alpha\sigma - i\delta}{4b\alpha^2\sigma}\right)^T,$$
$$P(b) = \left(\frac{\delta + (3\alpha^2 + b\alpha\sigma - 5)i}{2\delta}, -\frac{4i\alpha}{\delta}\right)$$

where

$$\delta = \sqrt{20b\alpha(1+\alpha^2)\sigma - (5 - 3\alpha^2 + b\alpha\sigma)^2}.$$

It is easy to check that

$$LQ = \lambda Q, \quad L^T P = \bar{\lambda} P, \quad \langle P, Q\rangle = \bar{P}_1 Q_1 + \bar{P}_2 Q_2 = 1.$$

Note that on l, $b = b^*$, namely $\sigma = 0$, $\omega = \omega_0$. The first Lyapunov coefficient ℓ_1 on l is given by

$$\ell_1\big|_{b=b^*} = \frac{1}{2\omega_0}\,\mathrm{Re}\, c_1\big|_{b=b^*}$$

where

$$c_1 = \big\langle P, C(Q, Q, \bar{Q}) + B(\bar{Q}, h_{20}) + 2B(Q, h_{11})\big\rangle$$

and h_{20} and h_{11} are given by

$$h_{20} = (2i\omega I_2 - A)^{-1} B(Q, Q), \;\; h_{11} = -A^{-1} B(Q, \bar{Q}).$$

First, we calculate

$$\begin{aligned} h_{20}|_{b=b^*} &= (4(25 + 15\alpha^4 + 5i\sqrt{5}\eta + \alpha^2\eta)/(15\alpha\eta^2), (-75 - 25\alpha^4 - 15i\sqrt{5}\eta \\ &\quad + \alpha^2(60 + 7i\sqrt{5}\eta))/(30\alpha^2(1+\alpha^2))^T, \\ h_{11}|_{b=b^*} &= (0, (5-\alpha^2)/(2\alpha^2))^T, \end{aligned}$$

where

$$\eta = \delta|_{b=b^*} = \sqrt{3\alpha^4 - 2\alpha^2 - 5}$$

Than we have

$$\ell_1|_{b=b^*} = \frac{(2\alpha^4 - 27\alpha^2 - 5)}{2\sqrt{5}\alpha^2(\alpha^2+1)\sqrt{(3\alpha^2-5)(\alpha^2+1)}}.$$

Thus we recover the result from [12] regarding the stability of limit cycle bifurcating from Hopf bifurcation.

Theorem 2.1 *Suppose that α satisfies (H1) and that b is sufficiently close to b^*. Then Sys.(2.1) exhibits a Hopf bifurcation as b crosses b^*. Moreover, if the assumption (H2) holds, the Hopf bifurcation is subcritical and hence the limit cycle bifurcating from the equilibrium point is stable as as b crosses b^* from left to right, and if the assumption (H3) holds the direction of the Hopf bifurcation is supercritical and the bifurcating periodic solutions are unstable.*

2.2 Second Lyapunov coefficient

If $\ell_1|_{b=b^*} = 0$, we have to consider Bautin bifurcation [4]. Clearly, $\ell_1|_{b=b^*} = 0$ if and only if $\alpha = \alpha^* \equiv \sqrt{\frac{27+\sqrt{769}}{4}} \approx 3.6990150461887854$. In this case we have

$$b^* = \frac{61 + 3\sqrt{769}}{2\sigma\sqrt{27+\sqrt{769}}} \approx \frac{9.745333956652765}{\sigma},$$
$$\omega_0 = \sqrt{\frac{5(\sqrt{769}-13)}{6}} \approx 3.5036706047282387.$$

and the second Lyapunov coefficient ℓ_2 at $b = b^*, \alpha = \alpha^*$ can be calculated by the following form

$$\begin{aligned} 12l_2(\alpha^*) = \frac{1}{12\omega_0}\operatorname{Re}\big\langle P, & E(Q,Q,Q,\bar{Q},\bar{Q}) + D(Q,Q,Q,\bar{h}_{20}) + 3D(Q,\bar{Q},\bar{Q},h_{20}) \\ & + 6D(Q,Q,\bar{Q},h_{11}) + C(\bar{Q},\bar{Q},h_{30}) + 3C(Q,Q,\bar{h}_{21}) + 6C(Q,\bar{Q},h_{21}) \\ & + 3C(Q,\bar{h}_{20},h_{20}) + 6C(Q,h_{11},h_{11}) + 6C(\bar{Q},h_{20},h_{11}) + 2B(\bar{Q},h_{31}) \\ & + 3B(Q,h_{22}) + B(\bar{h}_{20},h_{30}) + 3B(\bar{h}_{21},h_{20}) + 6B(h_{11},h_{21})\big\rangle \end{aligned}$$

where

$$\begin{aligned} h_{20} &= (2i\omega_0 - A)^{-1}B(Q,Q), \\ h_{11} &= -A^{-1}B(Q,\bar{Q}), \\ h_{21} &= (i\omega_0 - A)^{INV}[C(Q,Q,\bar{Q}) + B(\bar{Q},h_{20}) + 2B(Q,h_{11}) - 2c_1 Q], \\ h_{31} &= (2i\omega_0 - A)^{-1}[D(Q,Q,Q,\bar{Q}) + 3C(Q,Q,h_{11}) + 3C(Q,\bar{Q},h_{20}) \\ &\quad + 3B(h_{20},h_{11}) + B(\bar{Q},h_{30}) + 3B(Q,h_{21}) - 6c_1 h_{20}], \\ h_{22} &= -A^{-1}[D(Q,Q,\bar{Q},\bar{Q}) + 4C(Q,\bar{Q},h_{11}) + C(\bar{Q},\bar{Q},h_{20}) + C(Q,Q,\bar{h}_{20}) \\ &\quad + 2B(h_{11},h_{11}) + 2B(Q,\bar{h}_{20}) + 2B(\bar{Q},h_{21}) + B(\bar{h}_{20},h_{20}) - 4h_{11}(c_1 + \bar{c}_1)]. \end{aligned}$$

Let $\mu = (b, \alpha)$ and $\mu^* = (b^*, \alpha^*)$. We have

$$\begin{aligned} h_{20}|_{\mu=\mu^*} &= (-0.311357 + 0.0608385i, -0.652807 + 0.68949i)^T, \\ h_{11}|_{\mu=\mu^*} &= (0., -0.317288)^T, \\ h_{30}|_{\mu=\mu^*} &= (0.227942 - 0.0639309i, 0.48125 - 0.419928i)^T, \\ c_1|_{\mu=\mu^*} &= 0.0501026i, \\ h_{21}|_{\mu=\mu^*} &= (-0.0286001, -0.0696793 - 0.0994375i)^T, \\ h_{40}|_{\mu=\mu^*} &= (-0.263493 + 0.0836034i, -0.569289 + 0.432716i)^T, \\ h_{31}|_{\mu=\mu^*} &= (-0.0535563 - 0.0276558i, -0.183096 + 0.00250381i)^T, \\ h_{22}|_{\mu=\mu^*} &= (0, 0.202437 + 0.0464408i)^T. \end{aligned}$$

Based on the above calculations, we have

$$\ell_2(\mu^*) = -0.00593573 \neq 0.$$

2.3 Regularity

To study the regularity of the map $\mu = (b, \alpha) \to (\nu, \ell_1)$ near $\mu^* = (b^*, \alpha^*)$, we have to check if the determinant of the Jacobian Matrix of this map at $\mu = \mu^*$ is nonzero, namely,

$$\det \begin{pmatrix} \frac{\partial \nu}{\partial b} & \frac{\partial \nu}{\partial \alpha} \\ \frac{\partial \ell_1}{\partial b} & \frac{\partial \ell_1}{\partial \alpha} \end{pmatrix} \Bigg|_{\mu=\mu^*} \neq 0.$$

To do this, we have to calculate the first Lyapunov coefficient $\ell_1(b, \alpha)$ given by

$$\ell_1(b, \alpha) = \frac{\mathrm{Re}[c_1]}{\omega} - \sigma \frac{\mathrm{Im}[c_1]}{\omega^2} \tag{2.6}$$

where

$$c_1 = \frac{g_{21}}{2} + \frac{|g_{11}|^2}{\lambda} + \frac{|g_{02}|^2}{2(2\lambda - \bar{\lambda})} + \frac{g_{20} g_{11} (2\lambda + \bar{\lambda})}{2|\lambda|^2}.$$

We evaluate $g_{20}, g_{11}, g_{02}, g_{21}$ first:

$$\begin{aligned} g_{20} &= \langle P, B(Q, Q) \rangle \\ &= [12i\alpha^5 - 17ib\alpha^4\sigma - b(-5i+\delta)\sigma + b\alpha^2(20i + \delta)\sigma + \alpha(60i + 12\delta - ib^2\sigma^2) \\ &\quad + \alpha^3(-56i - 4\delta + ib^2\sigma^2)]/[(1+\alpha^2)^2\delta], \\ g_{02} &= \langle P, B(\bar{Q}, \bar{Q}) \rangle \\ &= [3i\alpha^4 + 5(5i + \delta) - \alpha^2(20i + \delta) - 5ib\alpha\sigma + 9ib\alpha^3\sigma]/[(\alpha + \alpha^3)\delta], \\ g_{11} &= \langle P, B(Q, \bar{Q}) \rangle \\ &= -(5i - 3i\alpha^2 + \delta + ib\alpha\sigma)(-5 - 16\alpha^2 + 5\alpha^4 + b\alpha\sigma - b\alpha^3\sigma)]/[2\alpha(1+\alpha^2)^2\delta], \\ g_{21} &= \langle P, C(Q, Q, \bar{Q}) \rangle \\ &= [-27i\alpha^6 - 3(5i + \delta) + 25ib\alpha^5\sigma - 2b\alpha^3(53i + \delta)\sigma + 3b\alpha(-i + 2\delta)\sigma \\ &\quad + \alpha^4(219i + 9\delta - 2ib^2\sigma^2) + \alpha^2(-281i - 58\delta + 6ib^2\sigma^2)]/[(1+\alpha^2)^3\delta]. \end{aligned}$$

Using Mathematica, we have

$$\frac{\partial(\nu, \ell_1)}{\partial(b, \alpha)}\Bigg|_{\mu=\mu^*} = \det \begin{pmatrix} \frac{\partial \nu}{\partial b} & \frac{\partial \nu}{\partial \alpha} \\ \frac{\partial \ell_1}{\partial b} & \frac{\partial \ell_1}{\partial \alpha} \end{pmatrix} \Bigg|_{\mu=\mu^*} = -0.00125023\sigma \neq 0.$$

Lemma 2.1 *The map $(b, \alpha) \to (\nu, \ell_1)$ is regular near $b = b^*, \alpha = \alpha^*$.*

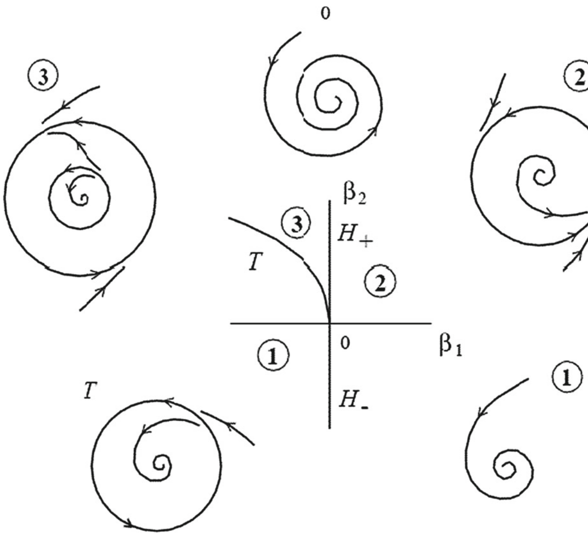

Fig. 1 Bautin bifurcation diagram

Now we give the normal form of Bautin bifurcation with unfolding parameters and its bifurcation diagram. Let $b = b^* + \tau_1, \alpha = \alpha^* + \tau_2$. Define

$$\beta_1 = \frac{\nu(b^* + \tau_1, \alpha^* + \tau_2)}{\omega(b^* + \tau_1, \alpha^* + \tau_2)}, \ \beta_2 = \sqrt{|\ell_2(\mu^*)|}\ell_1(b^* + \tau_1, \alpha^* + \tau_2).$$

Then after long calculation, we have

$$\begin{aligned}\beta_1 &= -0.0359523\sigma\tau_1 + 0.120995\tau_2 + \mathcal{O}(\|\tau\|^2),\\ \beta_2 &= -0.000188277\tau_1 + 0.00139831\tau_2 + \mathcal{O}(\|\tau\|^2).\end{aligned}$$

Then after performing nonlinear transforms, Sys.(2.5) is equivalent to the following truncated normal form [4]

$$\dot{z} = (\beta_1 + i)z + \beta_2 z|z|^2 - z|z|^4. \tag{2.7}$$

Theorem 2.2 *Sys.(2.1) exhibits Bautin bifurcation at $b = b^*, \alpha = \alpha^*$, around which Sys.(2.1) is equivalent to the normal form (2.7).*

The complete bifurcation diagram of Sys.(2.7) is shown in Fig. 1 from [4], where

$$\begin{gathered}H^+ = \{(\beta_1, \beta_2) : \beta_1 = 0, \beta_2 > 0\}, \ H^- = \{(\beta_1, \beta_2) : \beta_1 = 0, \beta_2 < 0\}\\ B^+ = \{(\beta_1, \beta_2) : \beta_1 > 0, \beta_2 = 0\}, \ B^- = \{(\beta_1, \beta_2) : \beta_1 < 0, \beta_2 = 0\},\\ T = (\beta_1, \beta_2) : \beta_2^2 + 4\beta_1 = 0, \beta_2 > 0\}.\end{gathered}$$

For (β_1, β_2) small enough,

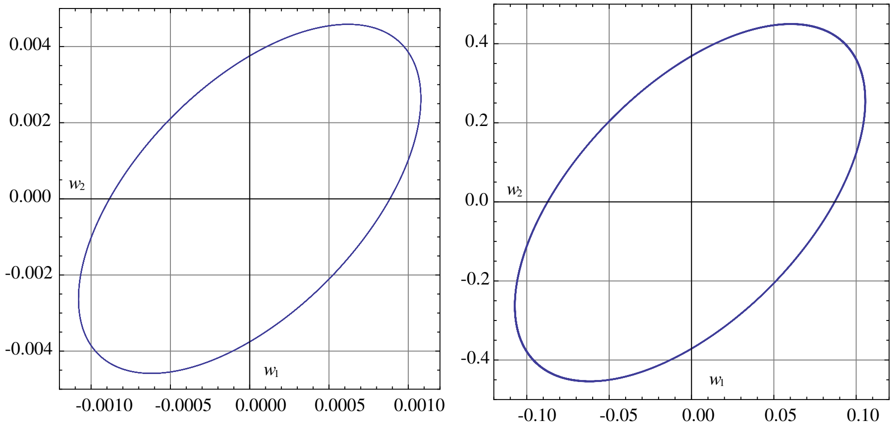

Fig. 2 Bautin bifurcation—*left*: A stable limit cycle when (τ_1, τ_2) is the region between B^+ and H^+, *right*: Two limit cycles when (τ_1, τ_2) is the region between T and H^+

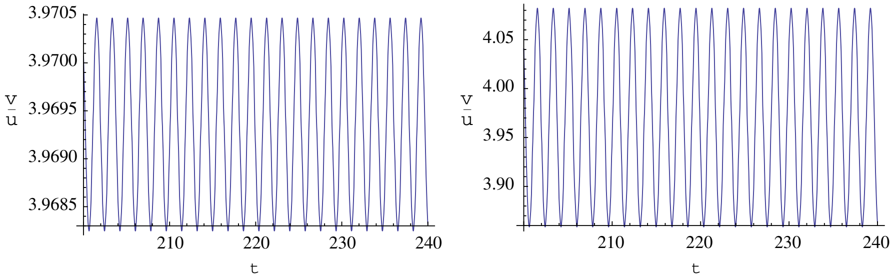

Fig. 3 Bautin bifurcation—*left*: The graph of the ratio of $\frac{[ClO_2^-]}{[I^-]}$ for Fig. 2(*left*); *right*: the graph of the ratio of $\frac{[ClO_2^-]}{[I^-]}$ for Fig. 2 (*right*)

(1) Between H^+ and B^+, there is a unique stable limit cycle bifurcating from (0,0),
(2) Between H^+ and T, there are two limit cycles of opposite stability which disappear and collide at the curve T.

Applying the above results and using the expressions of β_1, β_2, we obtain

$$\begin{aligned}
\overline{H}^+ &= \{(\tau_1, \tau_2) : \tau_2 = 0.297139\sigma\tau_1, \tau_1 > 0\},\\
\overline{H}^- &= \{(\tau_1, \tau_2) : \tau_2 = 0.297139\sigma\tau_1, \tau_1 < 0\}\\
\overline{B}^+ &= \{(\tau_1, \tau_2) : \tau_1 < 0, \tau_2 = 0.134646\sigma\tau_1\},\\
\overline{B}^- &= \{(\tau_1, \tau_2) : \tau_1 > 0, \tau_2 = 0.134646\sigma\tau_1\}\\
\overline{T} &= \{(\tau_1, \tau_2) : \tau_2 = 0.297139\sigma\tau_1 - 1.06672 \times 10^{-7}\sigma^2\tau_1^2 + \mathcal{O}(\tau_1^3), \tau_1 > 0\}
\end{aligned}$$

and the following theorem regarding the original Sys.(2.1).

Theorem 2.3 *For sufficiently small* τ_1, τ_2, *(i) Between $\overline{H}^+$ and $\overline{B}^+$, Sys.(2.1) has a unique stable limit cycle bifurcating from (0,0).*

(ii) Between $\overline{H}^+$ and $\overline{T}$, Sys.(2.1) has two limit cycles of opposite stability which disappear and collide at the curve $\overline{T}$.

3 Numerical simulations and conclusion

In this section, we give an example to verify (ii) of Theorem 2.3. Let $\sigma = 8$ and then $b^* = 1.2181667445815956$. Take $\tau_1 = 0.0001$, $\tau_2 = 0.00024$. Then $b = b^* + \tau_1 = 1.21827$, $\alpha = \alpha^* + \tau_2 = 3.69926$. It is easy to check that (τ_1, τ_2) is between $\overline{T}$ and $\overline{H}^+$. By Theorem 2.1, there are two limit cycle generated by Bautin bifurcation. Figure 2 verifies this result.

Laboratory observations have shown the formation of oscillations for reactions of CIMA as well as for CDIMA. It is also indicated that the oscillation depends on the ratio of $\frac{[ClO_2^-]}{[I^-]}$ ([1,2,7,9,11]). In this work we have demonstrated the existence of two limit cycles (Fig. 3). These two limit cycles indicate the formation of oscillations at different range of the inhibitor to activator ratio. A narrow range inhibitor (Fig. 3(left)) to activator ratio leads to the development of less stable oscillatory phenomenon which is associated with the smaller limit cycle. On the other hand, a wider range of inhibitor (Fig. 3(right)) to activator ratio leads to more stable and sustainable oscillatory cycle. This is consistent with the fact that the activator diffusion has to be slower than that of the inhibitor in order for the system to display oscillation.

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